Spatially Homogeneous Universes Which Admit Source-Free Magnetic Fields.

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In recent years interest in the Einstein-Cartan theory of space-time ([1-3]) has grown significantly. In the theory the space-time manifold is endowed with a torsion (non-symmetric connection) field, which couples to the spin density of matter in the universe.

Cosmological models based on the Einstein-Cartan theory are usually studied under the assumption that, having alligned the spins of the particles along a particular direction, the magnetic field that might be present in the universe decouples from the magnetic dipoles of the matter content ([4]).

Non coupling between matter and electromagnetic fields in the universe is also assumed in most cosmological models based on the theory of general relativity ([5,6]).

It is of interest, therefore, to determine the class of spatially homogeneous space-times which admit the presence of source-free magnetic fields. This can be done simultaneously in the Einstein-Cartan theory and in general relativity, because it has been shown ([2]), that the sourceless equations of Maxwell are the same in both theories, namely

$$\mathrm{d}F = 0 \tag{1}$$

and

$$\mathrm{d}{*F} = 0\,, \tag{2}$$

where F and $*F$ are the electromagnetic field 2-form and its dual, respectively.

In a spatially homogeneous space-time manifold, *i.e.* one which is locally invariant under a group of isometries G_3 simply transitive on spacelike surfaces, we can choose a set of basic 1-forms Θ^α ($\alpha = 0, 1, 2, 3$) with $\Theta^0 = \mathrm{d}t$, such that the hypersurfaces of homogeneity are given by

$$t(x^\alpha) = \text{constant}\,, \tag{3}$$

(1) F. W. HEHL: *Gen. Rel. and Gravit.*, **4**, 333 (1973).
(2) F. W. HEHL: *Gen. Rel. and Gravit.*, **5**, 491 (1974).
(3) A. TRAUTMAN: *Ann. N. Y. Accad. Sci.*, **262**, 241 (1975).
(4) A. K. RAYCHAUDHURI: *Phys. Rev.*, **12**, 952 (1975).
(5) D. R. BRILL: *Phys. Rev. Ser. B*, **133**, 845 (1964).
(6) D. G. TSOUBELIS: Ph. D. Dissertation, New York, N. Y., 1976 (unpublished).

where the x^α's are local co-ordinates. With respect to such a basis, we can write the magnetic-field 2-form as

$$F = \tfrac{1}{2}\varepsilon^i{}_{jk} B_i \Theta^j \times \Theta^k \,, \tag{4}$$

where $B_i = B_i(t)$, $\varepsilon^i{}_{jk}$ is the totally antisymmetric symbol ($\varepsilon^1{}_{23} = 1$), and Latin indices run from 1 to 3. The dual of F becomes

$$*F = B_i \Theta^0 \times \Theta^i \,. \tag{5}$$

In the above basis, on the other hand, we have (7)

$$\mathrm{d}\Theta^\alpha = -\tfrac{1}{2} C_{\beta\gamma}{}^\alpha \Theta^\beta \times \Theta^\gamma \,, \tag{6}$$

where $C_{\beta\gamma}{}^\alpha = C_{\beta\gamma}{}^\alpha(t) = C_{[\beta\gamma]}{}^\alpha$.

Substituting (4) and (5) into (1) and (2), respectively, and using (6), we obtain the following set of equations:

$$\frac{\mathrm{d}B^i}{\mathrm{d}t} + 2C_{0j}{}^{[i}B^{j]} = 0 \,, \tag{7}$$

$$B_l \varepsilon^l{}_{mn} C_{jk}{}^m \varepsilon^{jkn} = 0 \tag{8}$$

and

$$\varepsilon^{ijk} C_{jk}{}^l B_l = 0 \,. \tag{9}$$

One can now write the $C_{jk}{}^i$'s in terms of a symmetric relative tensor n^{ij} and a relative vector a_i as

$$C_{jk}{}^i = \varepsilon_{jkl} n^{li} + \delta^i_k a_j - \delta^i_j a_k \,, \tag{10}$$

where $n^{ij} = n^{ij}(t) = n^{(ij)}$, $\quad a_i = a_i(t)$, and

$$n^{ij} a_j = 0 \,. \tag{11}$$

When (10) is substituted in (8) and (9), they give

$$B^j a_j = 0 \tag{12}$$

and

$$(n^{ij} - \varepsilon^{ijk} a_k) B_j = 0 \,, \tag{13}$$

respectively.

Now, if $a_i = 0$ (class A space-times), it follows from (13) that $\det(n^{ij}) = 0$. In this case, when the rank of the matrix (n^{ij}) is equal to 2, 1 or zero, then the group types

(7) A. R. KING and G. F. R. ELLIS: *Commun. Math. Phys.*, **31**, 209 (1973).

is the Bianchi type VI_0 or VII_0, II and I, respectively. (See ([7]) for the Bianchi-Schücking-Behr classification of group types based on the solution of the characteristic equation of (n^{ij}).)

If $a_i \neq 0$ (class B space-times), then it follows from (11) that $\det(n^{ij}) = 0$, again. This, along with (12) and (13), implies that

$$[n^{ij} n_{ij} - (n_k{}^k)^2]/2a^k a_k = +1 . \tag{14}$$

(It can be easily proven that (14) holds, by using an orthonormal frame where $n^{ij} = \text{diag}(0, n^2, n^3)$, $a_i = (a, 0, 0)$, and $B^i = (0, B^2, B^3)$.) Thus, in this case the group type must be VI_{-1}, or Bianchi type III.

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